Dynamic effect of phase conjugation on wave localization

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Abstract

We investigate what would happen to the time dependence of a pulse reflected by a disordered single-mode waveguide, if it is closed at one end not by an ordinary mirror but by a phase-conjugating mirror. We find that the waveguide acts like a virtual cavity with resonance frequency equal to the working frequency ω_0 of the phase-conjugating mirror. The decay in time of the average power spectrum of the reflected pulse is delayed for frequencies near ω_0. In the presence of localization the resonance width is $\tau_s^{-1}\exp(-L/l)$, with L the length of the waveguide, l the mean free path, and τ_s the scattering time. Inside this frequency range the decay of the average power spectrum is delayed up to times $t \simeq \tau_s \exp(L/l)$.

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I. INTRODUCTION

The reflection of a wave pulse by a random medium provides insight into the dynamics of localization [1–4]. The reflected amplitude contains rapid fluctuations over a broad range of frequencies, with a slowly decaying envelope. The power spectrum $a(\omega, t)$ characterizes the decay in time t of the envelope at frequency ω. In an infinitely long waveguide (with N propagating modes), the signature of localization [5,6],

$$\Big\langle a(\omega, t) \Big\rangle \propto t^{-2} \quad \text{for} \quad t \gg N^2 \tau_s, \tag{1}$$

is a quadratic decay of the disorder-averaged power spectrum $\langle a \rangle$, that sets in after N^2 scattering times τ_s.

The decay (1) still holds over a broad range of times if the length L of the waveguide is finite, but much greater than the localization length $\xi = (N+1)l$ (with $l = c\tau_s$ the mean free path). What changes is that for exponentially large times $t \gg \tau_s \exp(L/l)$ the quadratic decay becomes more rapid $\propto \exp(-\text{constant} \times \ln^2 t)$. This is the celebrated log-normal tail [7–11]. We may assume that the finite length of the waveguide is realized by terminating one end by a perfectly reflecting mirror, so that the total reflected power is unchanged.

In this paper we ask the question what happens if instead of such a normal mirror one would use a phase-conjugating mirror [12,13]. The interplay of multiple scattering by disorder and optical phase conjugation is a rich problem even in the static case [14–16]. Here we show that the dynamical aspects are particularly striking. Basically, the disordered waveguide is turned into a virtual cavity with a resonance frequency ω_0 set by the phase-conjugating mirror.

We present a detailed analytical and numerical calculation for the single-mode case ($N = 1$). For times $t \gg \tau_s$ we find that $a(\omega, t)$ has decayed almost completely except in a narrow frequency range $\propto \tau_s^{-1} \exp(-L/l)$ around ω_0. In this frequency range the decay is delayed up to times $t \simeq \tau_s \exp(L/l)$, after which a log-normal decay sets in. The exponentially large difference in time scales for the decay near ω_0 and away from ω_0 is a signature of localization.

II. FORMULATION OF THE PROBLEM

A. Scattering theory

A scattering matrix formulation of the problem of combined elastic scattering by disorder and inelastic scattering by a phase-conjugating mirror was developed by Paasschens et al. [15]. We summarize the basic equations for the case of a single propagating mode in the geometry shown in Fig. 1. A single-mode waveguide is closed at one end ($x = 0$) by either a normal mirror or by a phase-conjugating mirror. Elastic scattering in the waveguide is due to random disorder in the region $0 < x < L$. For simplicity we consider a single polarization, so that we can use a scalar wave equation.

The phase-conjugating mirror is pumped at frequency ω_0. This means that a wave incident at frequency $\omega_0 + \omega$ will be retro-reflected at frequency $\omega_0 - \omega$, for $\omega \ll \omega_0$. For $x \gg L$ the wave amplitude at frequencies $\omega_\pm = \omega_0 \pm \omega$ is an incoming or outgoing plane wave,

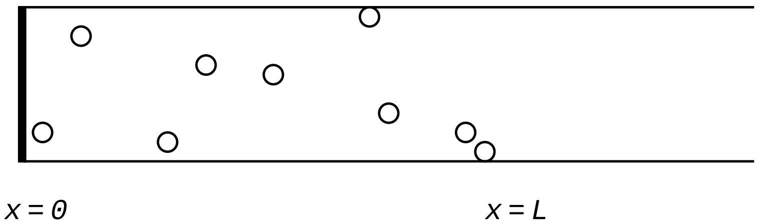

FIG. 1. The geometry under investigation consists of a single-mode waveguide with a mirror at $x = 0$. It can be a normal mirror or a phase-conjugating mirror. There are randomly positioned obstacles between $x = 0$ and $x = L$.

$$u_{\pm}^{\rm in}(\vec{r}, t) = \text{Re}\ \ \phi_{\pm}^{\rm in} \exp[-ik_{\pm}(x - L) - i\omega_{\pm}t]\psi_{\pm}(y, z), \tag{2a}$$

$$u_{\pm}^{\rm out}(\vec{r}, t) = \text{Re}\ \ \phi_{\pm}^{\rm out} \exp[ik_{\pm}(x - L) - i\omega_{\pm}t]\psi_{\pm}(y, z). \tag{2b}$$

Here $k_{\pm} = k_0 \pm \omega/c$ is the wavenumber at frequency $\omega_{\pm}$, with k_0 the wavenumber at ω_0 and $c = {\rm d}\omega/{\rm d}k$ the group velocity. The transverse wave profile $\psi_{\pm}(y, z)$ is normalized such that the wave carries unit flux.

The reflection matrix relates the incoming and outgoing wave amplitudes, according to

$$\begin{pmatrix} \phi_+ \\ \phi_-^* \end{pmatrix}^{\rm out} = \begin{pmatrix} r_{++} & r_{+-} \\ r_{-+} & r_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_-^* \end{pmatrix}^{\rm in}. \tag{3}$$

The reflection coefficients are complex numbers that depend on ω. They satisfy the symmetry relations

$$r_{--}^*(\omega) = r_{++}(-\omega), \qquad r_{-+}^*(\omega) = r_{+-}(-\omega). \tag{4}$$

If there is only reflection at the mirror, and no disorder, then one has simply

$$\begin{pmatrix} r_{++} & r_{+-} \\ r_{-+} & r_{--} \end{pmatrix} = \begin{pmatrix} -e^{2ik_+L} & 0 \\ 0 & -e^{-2ik_-L} \end{pmatrix} \tag{5}$$

for a normal mirror, and

$$\begin{pmatrix} r_{++} & r_{+-} \\ r_{-+} & r_{--} \end{pmatrix} = \begin{pmatrix} 0 & -ie^{2iL\omega/c} \\ ie^{2iL\omega/c} & 0 \end{pmatrix} \tag{6}$$

for a phase-conjugating mirror operating in the regime of ideal retro-reflection. (We will assume this regime in what follows.)

We wish to determine how the reflection coefficients are modified by the elastic scattering by the disorder. For this we need the elastic scattering matrix

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{7}$$

The reflection coefficients r,r' and transmission coefficients t,t' describe reflection and transmission from the left or from the right of a segment of a disordered waveguide of length

L. The matrix S is unitary and symmetric (hence $t = t'$). The basis for S is chosen such that $r = r' = 0$, $t(\pm\omega) = e^{ik_\pm L}$ in the absence of disorder. The relationship between the coefficients in Eqs. (3) and (7) is [15]

$$r_{++}(\omega) = r'(\omega) + t(\omega)[1 - r^*(-\omega)r(\omega)]^{-1}r^*(-\omega)t(\omega), \tag{8a}$$

$$r_{+-}(\omega) = -it(\omega)[1 - r^*(-\omega)r(\omega)]^{-1}t^*(-\omega), \tag{8b}$$

for a phase-conjugating mirror. For a normal mirror there is no mixing of frequencies and one has simply

$$r_{++}(\omega) = r'(\omega) - t(\omega)[1 + r(\omega)]^{-1}t(\omega), \tag{9a}$$

$$r_{+-}(\omega) = 0. \tag{9b}$$

In each case the matrix of reflection coefficients is unitary, so

$$|r_{++}(\omega)|^2 + |r_{+-}(\omega)|^2 = 1. \tag{10}$$

B. Power spectrum

We assume that a pulse $\propto \delta(t)$ is incident at $x = L$ [corresponding to $\phi_\pm^{\rm in} = 1$ for all ω in Eq. (2)]. The reflected wave at $x = L$ has amplitude

$$u_{\rm out}(t) = \mathrm{Re}\; e^{-i\omega_0 t}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\; \Big(\Big[r_{++}(\omega) + r_{+-}(\omega)\Big]e^{-i\omega t} + \Big[r^*_{--}(\omega) + r^*_{-+}(\omega)\Big]e^{i\omega t}\Big). \tag{11}$$

(We have suppressed the transverse coordinates y, z for simplicity of notation.) Using the symmetry relations (4), we can rewrite this as

$$u_{\rm out}(t) = \mathrm{Re}\; e^{-i\omega_0 t}\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi}\; \Big[r_{++}(\omega) + r_{+-}(\omega)\Big]e^{-i\omega t}. \tag{12}$$

The time correlator of the reflected wave becomes

$$\begin{aligned} u_{\rm out}(t)u_{\rm out}(t+t') = \tfrac{1}{2}\,\mathrm{Re}\; e^{i\omega_0 t'}\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi}\; e^{i(\omega'-\omega)t}e^{i\omega' t'} \\ \times\; \Big[r_{++}(\omega) + r_{+-}(\omega)\Big]\Big[r^*_{++}(\omega') + r^*_{+-}(\omega')\Big], \end{aligned} \tag{13}$$

plus terms that oscillate on a timescale $1/\omega_0$. We make the rotating wave approximation and neglect these rapidly oscillating terms. The power spectrum a of the reflected wave is obtained by a Fourier transform,

$$\begin{aligned} a(\omega, t) &= \int_{-\infty}^\infty \mathrm{d}t'\;\; \cos\,[(\omega_0 + \omega)t']u_{\rm out}(t)u_{\rm out}(t+t') \\ &= \mathrm{Re}\int_{-\infty}^\infty \frac{\mathrm{d}\delta\omega}{2\pi}\; e^{-i\delta\omega t}a(\omega, \delta\omega), \end{aligned} \tag{14}$$

where we have introduced the correlator in the frequency domain

$$a(\omega,\delta\omega)=\tfrac{1}{4}\Big[r_{++}(\omega+\delta\omega)+r_{+-}(\omega+\delta\omega)\Big]\Big[r^*_{++}(\omega)+r^*_{+-}(\omega)\Big]. \tag{15}$$

Integration of the power spectrum over time yields, using also Eq. (10),

$$\int_{-\infty}^{\infty}\mathrm{d}t\;\,a(\omega,t)=\mathrm{Re}\;\,a(\omega,\delta\omega=0)=\tfrac{1}{4}+\tfrac{1}{2}\,\mathrm{Re}\;\,r_{+-}(\omega)r^*_{++}(\omega). \tag{16}$$

For a normal mirror $r_{+-}(\omega)=0$ and $a(\omega,\delta\omega=0)=\frac{1}{4}$, expressing flux conservation. For the phase-conjugating mirror there is inelastic scattering, which mixes the frequency components ω and $-\omega$. The constraint of flux conservation then takes the form

$$a(\omega,\delta\omega=0)+a(-\omega,\delta\omega=0)=\tfrac{1}{2}. \tag{17}$$

This follows from the symmetry relations (4) and the unitarity of the reflection matrix. Eq. (17) implies that $a(\omega=0,\delta\omega=0)=\frac{1}{4}$.

III. RANDOM SCATTERERS

We assume weak disorder, meaning that the mean free path l is much larger than the wavelength $2\pi/k_0$. The multiple scattering by disorder localizes the wave with localization length equal to $2l$. We consider separately the case of a phase-conjugating mirror and of a normal mirror.

A. Phase-conjugating mirror

We first concentrate on the degenerate regime of small frequency shift ω, and will simplify the expressions by putting $\omega=0$ from the start. We note that $r_{++}(0)=0$, $r_{+-}(0)=-i$, as follows from Eq. (8) and unitarity of the scattering matrix (7). Using Eqs. (8) and (15), we arrive at the power spectrum in the frequency domain

$$a(0,\delta\omega)=\frac{i}{4}\Big(r'(\delta\omega)+\Big[1-r^*(-\delta\omega)r(\delta\omega)\Big]^{-1}\Big[t^2(\delta\omega)r^*(-\delta\omega)-it(\delta\omega)t^*(-\delta\omega)\Big]\Big). \tag{18}$$

The scattering amplitudes have the polar decomposition $r=\sqrt{R}\exp(i\theta)$, $r'=\sqrt{R}\exp(i\theta')$, $t=i\sqrt{1-R}\exp[\frac{1}{2}i(\theta+\theta')]$, with R,θ,θ' real functions of frequency. The phase θ' may be assumed to be statistically independent of $R(\pm\delta\omega),\theta(\pm\delta\omega)$, and uniformly distributed in $(0,2\pi)$. (This is the Wigner conjecture, proven for chaotic scattering in Ref. [17].) In this way only the last term in Eq. (18) survives the disorder average $\langle\cdots\rangle$,

$$4\Big\langle a(0,\delta\omega)\Big\rangle=\left\langle\frac{t(\delta\omega)t^*(-\delta\omega)}{1-r^*(-\delta\omega)r(\delta\omega)}\right\rangle=\sum_{m=0}^{\infty}Z_m, \tag{19}$$

where we have defined $Z_m=\langle t(\delta\omega)t^*(-\delta\omega)[r^*(-\delta\omega)r(\delta\omega)]^m\rangle$.

The moments Z_m satisfy the Berezinskii recursion relation [18,19]

$$l\frac{\mathrm{d}Z_m}{\mathrm{d}L}=m^2(Z_{m+1}+Z_{m-1}-2Z_m)+(2m+1)(Z_{m+1}-Z_m)+2i\tau_s\delta\omega(2m+1)Z_m, \tag{20}$$

with $\tau_s = l/c$ the scattering time. [The mean free path l accounts only for backscattering, so that the scattering time in a kinetic equation would equal $\frac{1}{2}\tau_s$.] The initial condition is $Z_m(L=0) = \delta_{m,0}$. In App. A we derive an analytical result for $\langle a(0,\delta\omega)\rangle$ in the small frequency range $\ln(1/\tau_s\delta\omega) \gtrsim L/l \gg 1$. It reads

$$\Big\langle a(0,\delta\omega)\Big\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k\ ik\,(-2i\tau_s\delta\omega)^{ik-1/2}2^{-3ik-1/2}\Gamma^2(\tfrac{1}{2}+ik)\Gamma(\tfrac{1}{2}-ik) \\ \times \Gamma^{-1}(1+ik)\Gamma^{-1}(ik)\exp[-(\tfrac{1}{4}+k^2)L/l]. \quad (21)$$

The initial decay is determined by the contributions of the poles at $k = -\frac{1}{2}i,\ -\frac{3}{2}i,\ -\frac{5}{2}i$,

$$\Big\langle a(0,\delta\omega)\Big\rangle = \tfrac{1}{4} + \tfrac{1}{4}i\tau_s\delta\omega\exp(2L/l) - \tfrac{1}{18}\tau_s^2\delta\omega^2\exp(6L/l) + \mathrm{O}(\delta\omega^3). \quad (22)$$

The result (21) is plotted in Fig. 2 for $L/l = 12.3$. We compare with the data from a numerical solution of the wave equation on a two-dimensional lattice, using the method of recursive Green functions [20]. (The method of simulation is the same as in Ref. [15], and we refer to that paper for a more detailed description.) The agreement with the analytical curves is quite good, without any adjustable parameter. The $\delta\omega$-dependence of $\langle a(0,\delta\omega)\rangle$ for large L/l occurs on an exponentially small scale, within the range of validity of Eq. (21).

A Fourier transform of Eq. (21) yields the average power spectrum in the time domain for $\ln(t/\tau_s) \gg L/l \gg 1$, with the result

$$\Big\langle a(0,t)\Big\rangle = \tfrac{1}{8}\pi^{3/2}(L/l)^{-3/2}\exp(-L/4l)\tau_s^{-1/2}t^{-1/2}\ln(4t/\tau_s)\exp\Big[-(l/4L)\ln^2(4t/\tau_s)\Big]. \quad (23)$$

The leading logarithmic asymptote of the decay is log-normal $\propto \exp[-(l/4L)\ln^2 t]$, characteristic of anomalously localized states [7–11].

These results are calculated for $\omega = 0$, and remain valid as long as $\omega \ll \tau_s^{-1}\exp(-L/l)$. This is the degenerate regime. For larger frequency mismatch ω one enters the non-degenerate regime. The power spectrum in that regime is the same as for a normal mirror, calculated in the next subsection.

B. Normal mirror

For comparison we discuss the known results for a disordered waveguide connected to a normal mirror instead of a phase-conjugating mirror. Since $r_{+-} = 0$, one has from Eq. (15)

$$4\Big\langle a(\omega,\delta\omega)\Big\rangle = \Big\langle r_{++}(\omega+\delta\omega)r^*_{++}(\omega)\Big\rangle \equiv R_1. \quad (24)$$

The quantities $R_m = \Big\langle [r_{++}(\omega+\delta\omega)r^*_{++}(\omega)]^m\Big\rangle$ satisfy the Berezinskii recursion relation [18,19]

$$l\frac{\mathrm{d}R_m}{\mathrm{d}L} = m^2(R_{m+1} + R_{m-1} - 2R_m) + 2i\tau_s\delta\omega m R_m. \quad (25)$$

The initial condition is $R_m(L=0) = 1$ for all m. The solution for $\ln(1/\tau_s\delta\omega) \gtrsim L/l$ is known [21], and gives the average power spectrum

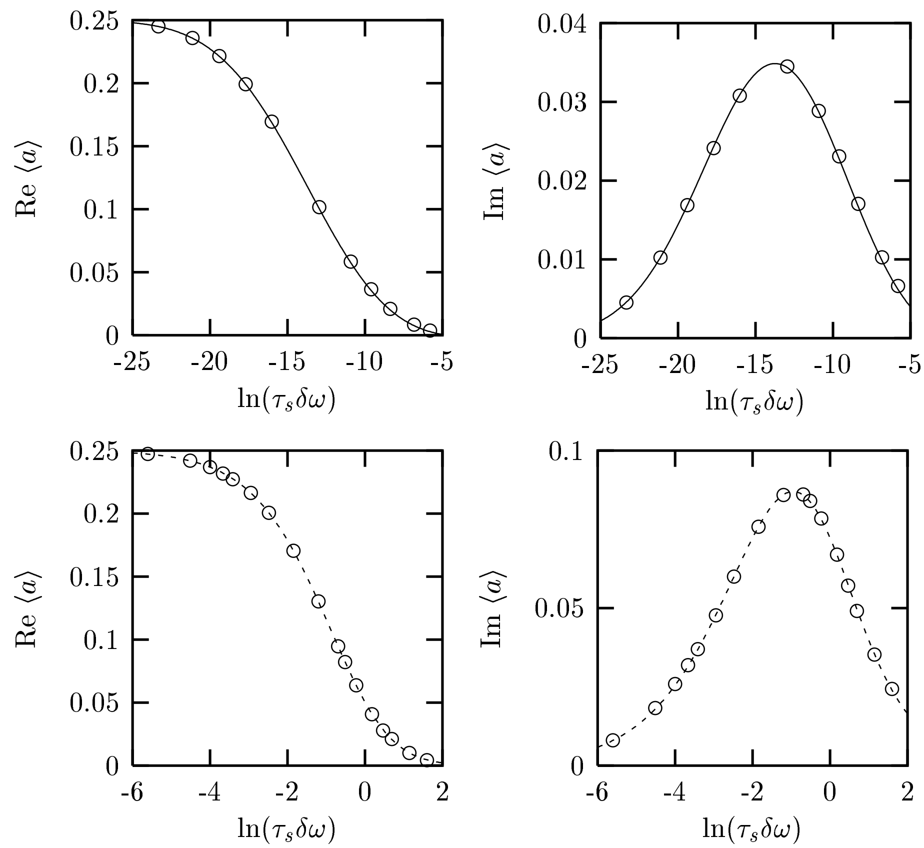

FIG. 2. Average power spectrum for reflection by a disordered waveguide ($L/l = 12.3$) connected to a phase-conjugating mirror [solid curves, from Eq. (21)] or a normal mirror [dashed curves, from Eq. (28)]. The data points follow from a numerical simulation. There is no adjustable parameter in the comparison. Notice the much faster frequency dependence for the phase-conjugating mirror (top panels), compared to the normal mirror (bottom panels).

$$\begin{aligned}\Big\langle a(\omega,\delta\omega)\Big\rangle &= \tfrac{1}{2}\sqrt{-2i\tau_s\delta\omega}\Big(\mathrm{K}_1\Big[2\sqrt{-2i\tau_s\delta\omega}\Big] \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}k\;\; k\sinh(\pi k)(\tfrac{1}{4}+k^2)^{-1}\mathrm{K}_{2ik}\Big[2\sqrt{-2i\tau_s\delta\omega}\Big]\exp\Big[-(\tfrac{1}{4}+k^2)L/l\Big]\Big), \qquad (26)\end{aligned}$$

with K a Bessel function. [The result (26) does not require $L/l \gg 1$, in contrast to Eq. (21).] The initial decay is dominated by the contributions of the poles at $k = -\frac{1}{2}i, -\frac{3}{2}i, -\frac{5}{2}i$,

$$\Big\langle a(\omega,\delta\omega)\Big\rangle = \tfrac{1}{4} + \tfrac{1}{2}i\tau_s\delta\omega L/l - \tfrac{1}{4}\tau_s^2\delta\omega^2\exp(2L/l) + \mathrm{O}(\delta\omega^3). \qquad (27)$$

Comparison of Eqs. (26) and (27) with Eqs. (21) and (22) shows that the decay is much slower for a normal mirror than for a phase-conjugating mirror. The characteristic frequency scale is larger by a factor $\exp(2L/l)$. So Eq. (26) is not sufficient to describe the entire decay of $\langle a(\omega,\delta\omega)\rangle$, which occurs in the range $\tau_s\delta\omega \lesssim 1$. The decay in this range is obtained by putting the left-hand-side of Eq. (25) equal to zero, leading to [5,22]

$$\Big\langle a(\omega,\delta\omega)\Big\rangle = \tfrac{1}{4} - \tfrac{1}{2}i\tau_s\delta\omega\exp(-2i\tau_s\delta\omega)\mathrm{Ei}(2i\tau_s\delta\omega), \qquad (28)$$

where Ei is the exponential integral function. The range of validity of Eq. (28) is $\ln(1/\tau_s\delta\omega) \ll L/l$ and $L/l \gg 1$. The result (28) is plotted in Fig. 2, and is seen to agree well with data from the numerical simulation.

For $\ln(t/\tau_s) \ll L/l$ (and $L/l \gg 1$) one can perform the Fourier transform of Eq. (28) to get the average power spectrum in the time domain [5],

$$\left\langle a(\omega,t)\right\rangle = \tfrac{1}{2}\tau_s(t+2\tau_s)^{-2}, \quad t>0. \tag{29}$$

It decays quadratically $\propto t^{-2}$ for $t/\tau_s \gg 1$. For exponentially long times, $t \gg \tau_s \exp(L/l)$, one should instead perform the Fourier transform of Eq. (26). One finds that the quadratic decay crosses over to a log-normal decay $\propto \exp[-(l/4L)\ln^2 t]$ [7], the same as for the phase-conjugating mirror.

IV. CONCLUSION

We have shown that the interplay of phase-conjugation and multiple scattering by disorder leads to a drastic slowing down of the decay in time t of the average power spectrum $\langle a(\omega,t)\rangle$ of frequency components ω of a reflected pulse. The slowing down exists in a narrow frequency range around the characteristic frequency ω_0 of the phase-conjugating mirror (degenerate regime). If ω is outside this frequency range (non-degenerate regime), the power spectrum decays as rapidly as for a normal mirror.

The slowing down can be interpreted in terms of a long-lived resonance at ω_0, that is induced in the random medium by the phase-conjugating mirror. This resonance is known from investigations of the static scattering properties [15]. The resonance is exponentially narrow $\propto \tau_s^{-1}\exp(-L/l)$ in the presence of localization (with τ_s the scattering time, L the length of the disordered region, and l the mean free path). The resonance leads to the exponentially large differences in time scales for the decay of the power spectrum in the degenerate regime and the non-degenerate regime.

We have restricted the calculation in this paper to the case of a single propagating mode, when a complete analytical theory could be provided. We expect that the N-mode case is qualitatively similar: An exponentially large difference in time scales $\propto \exp(L/\xi)$ for the decay in the degenerate and non-degenerate regimes provided the medium is localized [L large compared to the localization length $\xi = (N+1)l$]. In the diffusive regime we expect $\langle a(\omega,t)\rangle$ to decay on the time scale of the diffusion time $\tau_s(L/l)^2$. The difference with the non-degenerate regime (or a normal mirror) is then a factor $(L/l)^2$ instead of exponentially large.

In final analysis we see that phase conjugation greatly magnifies the difference in the dynamics with and without localization. Indeed, if there is no phase-conjugating mirror the main difference is a decay $\propto t^{-3/2}$ in the diffusive regime versus t^{-2} in the localized regime [6], but the characteristic time scale remains the same (set by the scattering time τ_s). We therefore suggest that phase conjugation might be a promising tool in the ongoing experimental search for dynamical features of localization [23,24].

APPENDIX A: POWER SPECTRUM IN THE FREQUENCY DOMAIN

We show how to arrive at the result (21) starting from the recursion relation (20). We assume $\ln(1/\tau_s\delta\omega) \gtrsim L/l \gg 1$. It is convenient to work with the Laplace transform

$$Z_m(\lambda) = \int_0^\infty \frac{\mathrm{d}L}{l}\ \exp(-\lambda L/l) Z_m(L) \tag{A1}$$

of the moments Z_m. The recursion relation (20) transforms into

$$\begin{aligned}\lambda Z_m(\lambda) - \delta_{m,0} = m^2\Big[Z_{m+1}(\lambda) + Z_{m-1}(\lambda) - 2Z_m(\lambda)\Big] + (2m+1)\Big[Z_{m+1}(\lambda) - Z_m(\lambda)\Big] \\ - \beta(2m+1)Z_m(\lambda),\end{aligned} \tag{A2}$$

with $\beta = -2i\tau_s\delta\omega$.

For small $|\beta|$ and large m this equation can be written as a differential equation,

$$m^2\frac{\partial^2 Z(m,\lambda)}{\partial m^2} + 2m\frac{\partial Z(m,\lambda)}{\partial m} - (\lambda + 2\beta m)Z(m,\lambda) = 0, \tag{A3}$$

where m is now considered to be a continuous variable. The solution of Eq. $(A3)$ is

$$Z(m,\lambda) = C(\lambda,\beta)(\beta m)^{-1/2}\mathrm{K}_{\sqrt{1+4\lambda}}\left(2\sqrt{2\beta m}\right). \tag{A4}$$

The factor $C(\lambda,\beta)$ is determined by matching to the solution of Eq. $(A2)$ for $\beta m \to 0$, $m \to \infty$, that has been calculated in Ref. [25]. The result is

$$\begin{aligned}C(\lambda,\beta) = 4\pi\beta^{1/2}\Gamma\Big(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\lambda}\Big)\Gamma^{-1}\Big(1 + \tfrac{1}{2}\sqrt{1+4\lambda}\Big)\Gamma^{-1}\Big(\tfrac{1}{2}\sqrt{1+4\lambda}\Big) \\ \times \exp\Big[\tfrac{1}{2}\sqrt{1+4\lambda}\ln(\beta/8)\Big].\end{aligned} \tag{A5}$$

To obtain the power spectrum (19) we replace the sum over m by an integration, with the result

$$\begin{aligned}\sum_{m=0}^{\infty} Z_m(\lambda) = 2^{1/2}\pi\beta^{-1/2}\Gamma^2\Big(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\lambda}\Big)\Gamma\Big(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\lambda}\Big)\Gamma^{-1}\Big(1 + \tfrac{1}{2}\sqrt{1+4\lambda}\Big) \\ \times \Gamma^{-1}\Big(\tfrac{1}{2}\sqrt{1+4\lambda}\Big)\exp\Big[\tfrac{1}{2}\sqrt{1+4\lambda}\ln(\beta/8)\Big].\end{aligned} \tag{A6}$$

There are poles at $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$ and a branch cut starting at $\lambda = -1/4$. When doing the inverse Laplace transform we put the integration path in between the poles and the branch cut. The final result is given by Eq. (21). The reason that we need the condition $L/l \gg 1$ is that Eqs. $(A4)$ and $(A5)$ are only correct for $m \gg 1$. The first terms in the sum $\sum_{m=0}^{\infty} Z_m$ are important for $L/l \lesssim 1$, but can be neglected for $L/l \gg 1$.

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